Trapped Surfaces Due to Concentration of Gravitational Radiation

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Sequences of nonsingular, asymptotically flat initial data for general relativity (GR) *in vacuo*, called critical sequences, are defined which approach the strong-field limit of GR in a precise sense. It is proven that critical sequences contain trapped surfaces for large values of the argument. Thus, by a theorem due to Penrose, the spacetimes evolving from all such configurations must develop singularities. In the course of the proof a new and conceptually simple proof of the positivity of the Arnowitt-Deser-Misner mass in the strong-field regime is obtained.

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It has long been believed—and recently proved by Christodoulou and Klainerman[1]—that initially regular, asymptotically flat solutions to the Einstein vacuum equations do not develop singularities provided the initial data are sufficiently weak in a precise sense. In this Letter we consider the opposite extreme of very strong fields. We define and study certain sequences of initial data *in vacuo* which are asymptotically flat and everywhere regular, but do not satisfy the weakness assumption. We show that, for large values of the parameter, the elements of these sequences contain trapped surfaces: A trapped surface (TS) is a closed two-surface such that pencils of light orthogonal to it, whether outgoing or ingoing, locally decrease in area.[2] It then follows from a famous theorem of Penrose[3] that the spacetimes evolving from these data are singular. The dynamical problem of whether these singularities are hidden inside a black hole remains open.[4] But it is the "pure" case of complete vacuum considered here which perhaps offers the best chances of attacking this question (e.g., no equation of state to worry about).

For simplicity, here we treat the time-symmetric case, where the only variable is the initial Riemannian metric g' on a three-manifold $\tilde{M}$ which, by virtue of Einstein's equations, has to have zero scalar curvature. The standard way of solving this constraint (see, e.g., Choquet-Bruhat and York[5]) is to pick some metric $\tilde{g}$ on $\tilde{M}$ having fast decay at infinity. One now looks for a metric g' in the conformal class defined by $\tilde{g}$ with zero scalar curvature. In other words, we try to find a positive function ϕ, tending to 1 at infinity, so that $g'=\phi^4\tilde{g}$ satisfies

$$R[g']=0 \Longleftrightarrow L_{\tilde{g}}\phi \equiv (-\Delta_{\tilde{g}}+\tfrac{1}{8}R[\tilde{g}])\phi=0\,, \qquad \phi>0,\quad \phi\rightarrow 1 \text{ at } \infty\,, \tag{1}$$

where R is the scalar curvature, $\Delta_{\tilde{g}}=\tilde{g}^{ab}\tilde{D}_a\tilde{D}_b$ is the Laplacian and $L_{\tilde{g}}$ is (minus) the conformal Laplacian of a metric $\tilde{g}$. The (conformal) metric $\tilde{g}$, called the "free data," is in fact not completely free, but has to satisfy a global inequality in order for (1) to have a solution. We now consider an infinite sequence of metrics $\tilde{g}_n$ on which we can solve (1), but tending as $n\rightarrow\infty$ to a metric $\tilde{g}_\infty$ which violates this global inequality. We call this a critical sequence (CS). We show that the Arnowitt-Deser-Misner (ADM) mass m_n, which is basically the monopole (Schwarzschild) part of the solution ϕ_n, grows unboundedly as n increases. (In particular, it must be positive for large n.) In addition, we prove that the higher-order multipoles of ϕ_n diverge no more rapidly than m_n, so that, loosely speaking, the geometry at fixed large radii is dominated by its Schwarzschildian contribution even in the limit as $n\rightarrow\infty$. Consequently, along any CS for large n, the existence of TS's can be inferred from the properties of the maximally extended Schwarzschild geometry, namely, that the mean curvature of the surfaces of constant radius on a $t=$const slice changes sign at the throat.

For technical reasons we work on a "conformally compactified" manifold M. Let M be a compact three-manifold, and g a positive-definite metric on M. Let Λ be a point of M, the "point at infinity," kept fixed throughout. For L_g as in Eq. (1), we look for a positive function G which is smooth on $\tilde{M}=M\backslash\Lambda$, but blows up at Λ in such a way that

$$\int_M (L_g f)G\,dV_g=4\pi f|_\Lambda,\quad G>0\,, \tag{2}$$

for all smooth functions f on M. In particular, $L_gG=0$ outside Λ. In other words, G is a global Green function for L_g with respect to the point Λ. Let $\Omega^{1/2}$ be an asymptotic distance function (ADF) near Λ, i.e., Ω satisfies ($\Omega_a\equiv D_a\Omega$, $\Omega_{ab}\equiv D_aD_b\Omega$)

$$\Omega|_\Lambda=0=\Omega_a|_\Lambda,\quad (\Omega_{ab}-2g_{ab})|_\Lambda=0,\quad \Omega_{abc}|_\Lambda=0\,, \tag{3}$$

and is extended as a smooth, positive but otherwise arbitrary function to all of M. Note that[6]

$$G=\Omega^{-1/2}+m/2+O(\Omega^{1/2}),\quad m=\text{const}\,, \tag{4}$$

where m does not depend on the choice of ADF. Next recall that under conformal transformations $\bar{g}=\omega^2 g$ ($\omega>0$) we have the operator equation

$$L_{\bar{g}}\circ\omega^{-1/2}=\omega^{-5/2}\circ L_g\,, \tag{5}$$

so that, given an initial-data set (g,G), there exists a whole class, given for each ω by

$$\bar{g}=\omega^2 g,\quad \bar{G}=\omega^{-1/2}G,\quad \Longrightarrow \bar{m}=m\,, \tag{6}$$

provided that the "scaling" is fixed, i.e., $\omega|_\Lambda=1$. The connection of the present setting with the standard one is as follows: The manifold $(\tilde{M},\tilde{g})$ with $\tilde{g}=\Omega^{-2}g$, $\Omega^{1/2}$ being some ADF, is asymptotically flat near Λ with zero ADM mass.[7] However, the metric $g'=G^4 g$, due to Eq. (4), is asymptotic to the Schwarzschild metric of mass m.[8] We also have

$$R[g']=G^{-4}(R[g]-8G^{-1}\Delta_g G)=0 \text{ on } \tilde{M}\,, \tag{7}$$

and $\phi=\Omega^{1/2}G$ satisfies the Lichnerowicz equation (1) with respect to $\tilde{g}$ on the "physical" manifold $\tilde{M}$. Note that the physical metric g' depends only on the conformal class of g provided that $\omega|_\Lambda=1$.

Now recall that L_g, defined on $C^\infty(M)$, is an essentially self-adjoint operator on $L^2(M)$ (with the standard inner product). Its spectrum consists of real isolated eigenvalues which are bounded below. The lowest eigenvalue $\lambda_1(g)$ is known[9] to be nondegenerate and the associated eigenfunction nowhere zero.

The following result has been effectively proven by Cantor and Brill.[10]

Theorem.— There exists a unique positive solution G of Eq. (2) if and only if $\lambda_1(g)>0$. (This restricts the possible topologies of M;[11] $M\cong S^3$, $\tilde{M}\cong\mathbf{R}^3$ is permitted.)

Proof.— The "if" direction is essentially given by Lee and Parker.[6] Conversely, let there be a positive G satisfying (1). Applying Eq. (1) to a solution $f_1(g)$ of $L_g f_1=\lambda_1(g)f_1$, chosen to be positive, we have that

$$\lambda_1\int_M f_1 G\,dV_g=4\pi f_1|_\Lambda>0\,. \tag{8}$$

Since $f_1>0$ and $G>0$ it follows that $\lambda_1(g)>0$.

We should point out that the sign of $\lambda_1(g)$ coincides with that of the conformally invariant Yamabe number of g,[12] whereas the *value* of $\lambda_1(g)$, due to its lack of conformal invariance, is physically uninteresting.

A critical sequence $\{g_n\}$ is now defined as a sequence of metrics with $\lambda_1(g_n)>0$ and converging uniformly, with derivatives of at least three orders, to a metric g_∞ for which $\lambda_1(g_\infty)=0$. We also assume the CS to be such that the metrics g_n and connections ∂g_n all coincide at the point Λ, so that an ADF $\Omega^{1/2}$ satisfying (3) for $g=g_n$ can be chosen independently of n.

Lemma (Ó Murchadha).— $\limsup_{n\to\infty}\max_M\phi_n=\infty$, where $\phi_n=\Omega^{1/2}G_n$.[13]

Proof.— Again by (2) we have, writing $L_n=L_{g_n}$, etc.,

$$\int_M (L_n f_1)G_n\,dV_n=4\pi f_1|_\Lambda\,, \tag{9}$$

where we choose $f_1=f_1(g_\infty)$, satisfying $L_\infty f_1=\lambda_1(g_\infty)f_1=0$ and $f_1>0$. Thus

$$0<4\pi f_1|_\Lambda\le(\max_M\phi_n)\left(\int_M\Omega^{-1/2}dV_n\right)\max_M|L_n f_1|\,. \tag{10}$$

The second factor on the right-hand side of (10) is bounded and the third one goes to zero as $n\to\infty$, which proves the lemma.

The next step shows that the point at which $\max\phi_n$ is achieved stays away uniformly from Λ. First note that the CS $\{g_n\}$ can be conformally rescaled, without changing its defining properties, so that $R[g_n]\ge 0$ in some n-independent neighborhood of Λ. Having done so, consider the formula

$$\Omega^{-2}R[\tilde{g}]=R[g]+2\Omega^{-2}(\Omega\Delta\Omega-3\Omega_a\Omega^a)\equiv R[g]+2\sigma\,. \tag{11}$$

From (3), σ at Λ is direction dependent but finite. A change of ADF $\Omega'=\mu\Omega$ ($\mu|_\Lambda=1$, $\mu_a|_\Lambda=0$) can be made with the result that, for all n, and for all directions, $\lim_{\to\Lambda}\sigma_n\ge s>0$. Then there is an n-independent punctured open neighborhood N of Λ, such that

$$R[\tilde{g}_n]\ge 0 \text{ in } \tilde{N},\quad \tilde{g}_n=\Omega^{-2}g_n\,. \tag{12}$$

Hence $L_{\tilde{g}_n}\phi_n=0$ admits a maximum principle on $\tilde{N}$ (see Ref. 14). Taking a value for n large enough so that $\max_{\tilde{M}}\phi_n>1$ it follows that this maximum may be obtained neither in $\tilde{N}$ nor at Λ (where $\phi_n\to 1$). Thus, making $\tilde{N}$ smaller if necessary, $\max\phi_n$ is obtained in $\mathcal{C}=\tilde{M}\backslash\tilde{N}$ for large n.[15]

In the bounded set $\mathcal{C}$ we can, since $\phi_n>0$, invoke the *Harnack inequality* (see Ref. 14) which implies that

$$\max_{\mathcal{C}}\phi_n\le C_n\min_{\mathcal{C}}\phi_n\le C_n\min_{\partial\mathcal{C}}\phi_n,\quad C_n>0\,, \tag{13}$$

where C_n is uniformly bounded when $\tilde{g}_n$, $\partial\tilde{g}_n$, $\partial\partial\tilde{g}_n$ are, so that C_n can be chosen to be n independent: $C_n\le C$. Consequently $\min_{\partial\mathcal{C}}\phi_n$ blows up as $n\to\infty$. Hence $\min_{\partial\mathcal{C}}G_n$ also blows up.

Now consider G_n. It is known that for an elliptic equation there always exists in a sufficiently small neighborhood of Λ a local fundamental solution[16] G^{loc} of the form

$$G^{\text{loc}}=\Omega^{-1/2}V+W\,, \tag{14}$$

where V,W can be chosen so that

$$V=1+O(\Omega),\quad W|_\Lambda=0\,. \tag{15}$$

Doing this for each n we obtain regular functions F_n

$=G_n - G_n^{\mathrm{loc}}$ in $N=\tilde{N}\cup\Lambda$ obeying

$$L_n F_n = 0 \text{ in } N. \tag{16}$$

Since $R[g_n]\geq 0$, F_n satisfies a minimum principle. Equation (16) tells us that an interior minimum of F_n, if it occurs, has to be positive. Since G_n^{loc} remains finite and bounded on $\partial\mathscr{C}$, whereas $\min_{\partial\mathscr{C}} G_n$ blows up as $n\to\infty$, we have that $\min_{\partial\mathscr{C}} F_n$ must become positive for large n. Thus $F_n>0$ for large n, so that $m_n = 2F_n|_\Lambda$ is also positive for large n. This is our positive-mass theorem.

We can now apply the Harnack inequality to F_n in N with the result that

$$\max_{\partial\mathscr{C}} F_n \leq \sup_N F_n \leq D \inf_N F_n, \quad D>0. \tag{17}$$

But if $\min_{\partial\mathscr{C}} F_n\to\infty$, then clearly $\max_{\partial\mathscr{C}} F_n\to\infty$ as $n\to\infty$. So, according to (17), F_n blows up at all points of N and in particular $m_n\to\infty$ as $n\to\infty$. Now define the *renormalized sequence* $\bar{F}_n$ in N by

$$\bar{F}_n \equiv F_n/(F_n|_\Lambda) = (2/m_n)F_n. \tag{18}$$

Since $\bar{F}_n \leq \sup_N F_n/\inf_N F_n$, (17) implies that $\bar{F}_n$ stays uniformly bounded as $n\to\infty$. But the *Schauder inequality*[14] shows that

$$\sup_N|\partial\bar{F}_n| + \sup_N|\partial\partial\bar{F}_n| \leq E\sup_N \bar{F}_n \tag{19}$$

for some constant $E>0$. In particular, $\partial\bar{F}_n$ and $\partial\partial\bar{F}_n$ are also uniformly bounded.

We now come to the issue of TS's. For sufficiently small $\Omega>0$ the level sets of Ω are smooth surfaces $\cong S^2$, which enclose a compact region in $\tilde{M}$. Since g_n' are time-symmetric initial data on $\tilde{M}$, the condition for $\Omega=$const to be outer trapped is that H_n, the trace of its extrinsic curvature with respect to the metric $g_n' = G_n^4 g_n$, with the normal pointing towards infinity, be negative. A computation gives[17]

$$-H_n = \frac{G_n^{-3}\Omega^{-1/2}}{(\Omega_c\Omega^c)^{1/2}}\left[\Omega^{1/2}G_n\left(g_n^{ab} - \frac{\Omega^a\Omega^b}{\Omega_d\Omega^d}\right)\Omega_{ab} + 4\Omega^{1/2}G_n^a\Omega_a\right]. \tag{20}$$

From the previous analysis of F_n and the Taylor theorem we have that G_n satisfies

$$G_n = \Omega^{-1/2} + m_n/2 + m_n O(\Omega^{1/2}),$$
$$(G_n - \Omega^{-1/2})_a = m_n O(1), \tag{21}$$

where the constants involved in the O symbols are understood to be independent of n. Using (21) and (3) in Eq. (20), we infer

$$H_n = \frac{4G_n^{-3}\Omega^{-1/2}}{(\Omega_c\Omega^c)^{1/2}}\left[1 - \frac{m_n\Omega^{1/2}}{2} + m_n O(\Omega)\right], \tag{22}$$

which immediately gives the following theorem.

Theorem.—Positive constants ε and δ exist which are independent of n, so that the surfaces $\Omega=\Omega_0$ with

$$\varepsilon \leq \Omega_0^{-1/2} \leq m_n/2 - \delta \tag{23}$$

are (outer) trapped for large n. Thus every sufficiently far-out surface of constant Ω gets trapped eventually and the area of the outermost trapped surface diverges like $O(m_n^2)$.

To avoid confusion we remark that, although the quantities appearing in (23) depend on scale, i.e., change under replacing g_n by $c^2 g_n$ ($c^2=$const$\neq 1$), the theorem itself is true for any scaling. Finally, we point out that several methods are available in the literature[18] allowing one to obtain large classes of CS's. We wish to stress that this is the first rigorous proof of the fact that pure gravitational waves can undergo gravitational collapse.

It seems reasonable to conjecture that, in some suitable norm, the boundary of the set where $\lambda_1(g)>0$ consists only of metrics with $\lambda_1(g)=0$. With such a result one could combine our proof of the strong-field positivity of mass with the weak-field proof of Brill and Deser[19] to obtain a full proof of the positivity of mass. Details and generalizations of the present work will be published elsewhere.

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[1]D. Christodoulou and S. Klainerman, Ann. Math. (to be published).

[2]Strictly speaking, these surfaces are *outer trapped,* in that only outgoing beams of light locally decrease in area. Inspection of Penrose's proof [R. Penrose, Phys. Rev. Lett. **14**, 57 (1965); R. Penrose, in *Battelle Rencontres,* edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), p. 121] shows that, with "trapped" being replaced by "outer trapped," the theorem goes through as before.

[3]Penrose (Ref. 2).

[4]See, however, W. Israel, Phys. Rev. Lett. **56**, 789 (1986).

[5]Y. Choquet-Bruhat and J. W. York, Jr., in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), Vol. 1, p. 99.

[6]See, e.g., J. M. Lee and T. H. Parker, Bull. Am. Math. Soc. **17**, 37 (1987).

[7]To see this, take coordinates x^a centered at Λ with $g_{ab}|_\Lambda = \delta_{ab}$ and $\partial g_{ab}|_\Lambda = 0$ and use "Kelvin-transformed" coordinates $\tilde{x}^a = \Omega^{-1}x^a$.

[8]This decay is physically reasonable at least for data with zero momentum.

[9]J. L. Kazdan, International Center for Theoretical Physics

Report No. SMR. 404/14, 1990 (unpublished).

[10]M. Cantor and D. Brill, Compositio Math. **43**, 317 (1981); compare also D. Brill, Ann. Phys. (N.Y.) **7**, 466 (1959).

[11]R. Schoen and S.-T. Yau, Ann. Math. **110**, 127 (1979); M. Gromov and H. B. Lawson, Jr., Inst. Hautes Etudes Sci. Publ. Math. **58**, 83 (1983).

[12]See Ref. 6 and N. Ó Murchadha, in *Proceedings of the Centre for Mathematical Analysis,* edited by R. Bartnik (Australian National University, Canberra, 1989), Vol. 19, p. 137.

[13]N. Ó Murchadha, Class. Quantum Grav. **4**, 1609 (1987).

[14]D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of the Second Order* (Springer, Berlin, 1983).

[15]It is impossible for large n to choose $R[\tilde{g}_n] \geq 0$ in $\mathcal{C}$ as well, as the maximum principle would otherwise give a contradiction. Since $\tilde{g}_n$ has zero mass, this is the starting point for an alternative derivation of a positive-mass result along the lines of R. Schoen and S.-T. Yau, Commun. Math. Phys. **90**, 575 (1983). However, we choose a different route.

[16]P. Garabedian, *Partial Differential Equations* (Wiley, New York, 1964).

[17]Note that Ω^a, Ω_{ab} depend implicitly on g_n^{ab}.

[18]J. L. Kazdan and F. W. Warner, Proc. Symp. Pure Math. **27**, 219 (1975); H. Eliasson, Math. Scand. **29**, 317 (1971); and the first citation of Ref. 10.

[19]D. Brill and S. Deser, Ann. Phys. (N.Y.) **50**, 548 (1968); Y. Choquet-Bruhat and J. Marsden, Commun. Math. Phys. **51**, 283 (1976).

[20]J. A. Wheeler, *Relativity, Groups and Topology,* edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964).